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# Real and integer domination in graphs

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**Abstract**

For an arbitrary subset $\mathscr{P}$ of the reals, we define a function $f: V \to \mathscr{P}$ to be a $\mathscr{P}$-dominating function of graph $G=(V,E)$ if the sum of the function values over any closed neighbourhood is at least 1. That is, for every $v \in V$, $f(N(v) \cup \{v\}) \geqslant 1$. The $\mathscr{P}$-domination number of a graph is defined to be the infimum of $f(V)$ taken over all $\mathscr{P}$-dominating functions $f$. When $\mathscr{P}=\{0,1\}$ one obtains the standard domination number. We obtain various theoretical and computational results on the $\mathscr{P}$-domination number of a graph. 

*Keywords:* Domination; Integer domination; Real domination

## 1. Introduction

All our graphs are finite and without loops or multiple edges. For a graph $G=(V,E)$ with vertex set $V$ and edge set $E$, the open neighbourhood of $v \in V$ is $N(v)=\{u \in V \mid uv \in E\}$ and the closed neighbourhood of $v$ is $N[v]=\{v\} \cup N(v)$. For a set $S$ of vertices, we define the open neighbourhood $N(S)=\bigcup_{v \in S} N(v)$, and the closed neighbourhood $N[S]=N(S) \cup S$.

A *dominating set* $S \subseteq V$ for a graph $G=(V,E)$ is such that each $v \in V$ is either in $S$ or adjacent to a vertex of $S$. (That is, $N[S]=V$.) The *domination number* of $G$, $\gamma(G)$, equals the minimum cardinality of a dominating set.

For a real-valued function $f: V \to \mathbf{R}$ the *weight* of $f$ is $w(f)=\sum_{v \in V} f(v)$, and for $S \subseteq V$ we define $f(S)=\sum_{v \in S} f(v)$, so $w(f)=f(V)$. For a vertex $v$ in $V$, we denote $f(N[v])$ by $f[v]$. Let $f: V \to \{0,1\}$ be a function which assigns to each vertex of a graph an element of the set $\{0,1\}$. We say $f$ is a *dominating function* if for

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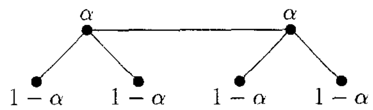


Fig. 1. A tree $T$ with $\gamma_{\mathbf{Z}} = -\infty$.

every $v \in V$, $f[v] \geqslant 1$. Then the domination number of a graph $G$ can be defined as $\gamma(G) = \min\{w(f) \mid f$ is a dominating function on $G\}$.

Several authors have suggested changing the allowable weights. Recently, Bange et al. [1] introduced the generalisation to $\mathscr{P}$-domination for an arbitrary subset $\mathscr{P}$ of the reals $\mathbf{R}$. A function $f : V \to \mathscr{P}$ is a $\mathscr{P}$-dominating function (or simply a dominating function) if the sum of its function values over every closed neighbourhood is at least 1. That is, for every $v \in V$, $f[v] \geqslant 1$. The $\mathscr{P}$-domination number of a graph $G$, denoted $\gamma_{\mathscr{P}}(G)$, is defined to be the infimum of $w(f)$ taken over all $\mathscr{P}$-dominating functions $f$.

Of course, this value might be $-\infty$. For example, if $\mathscr{P} = \mathbf{Z}$ (where $\mathbf{Z}$ denotes the integers) and $\alpha$ is a positive integer, then for the tree $T$ shown in Fig. 1, $\gamma_{\mathscr{P}}(T) \leqslant 4 - 2\alpha$. As we can make $\alpha$ as large as we like, it is evident that $\gamma_{\mathbf{Z}}(T) = -\infty$.

When $\mathscr{P} = \{0, 1\}$ we obtain the standard domination number. When $\mathscr{P} = [0, 1]$, we obtain the *fractional domination number*, denoted $\gamma_f(G)$, introduced by Hedetniemi et al. [19]. When $\mathscr{P} = \{-1, 0, 1\}$ we obtain the *minus domination number*, and when $\mathscr{P} = \{-1, 1\}$ we obtain the *signed domination number*, introduced by Dunbar et al. [10,11]. For a survey of these parameters see [21].

A trivial observation is that if $\mathscr{P} \subseteq \mathscr{Q}$, then $\gamma_{\mathscr{P}}(G) \geqslant \gamma_{\mathscr{Q}}(G)$. However, unlike the standard domination number the $\mathscr{P}$-domination number is not necessarily monotonic. For example, if one removes the central edge from the tree $T$ in Fig. 1 then the resultant graph has $\gamma_{\mathbf{Z}} = 2$.

## 2. Real domination

In this section we show that there is a simple solution to the $\mathbf{R}$-domination number of a graph.

The relationship between domination and linear programming has been exploited by several authors. For, the concept of domination can be formulated in terms of solving a $\{0, 1\}$-integer programming problem. For $V = \{v_1, v_2, \ldots, v_n\}$ we define the *closed neighbourhood matrix* to be $N = [n_{i,j}]$ where $1 \leqslant i,\ j \leqslant n$, and $n_{i,j} = 1$ if $i = j$ or if $v_i v_j \in E$, and $n_{i,j} = 0$ otherwise. For $S \subseteq V$, we let $\boldsymbol{x}(S) = (x_1, x_2, \ldots, x_n)^t$ be the column vector with $x_i = 1$ if $v_i \in S$, and $x_i = 0$ if $v_i \notin S$. Then $S$ is a dominating set if and only if $N\boldsymbol{x}(S) \geqslant \mathbf{1}$, where $\mathbf{1}$ denotes the all 1's column vector in $\mathbf{R}^n$. So $\gamma(G) = \min \sum_{i=1}^{n} x_i$ subject to $N\boldsymbol{x} \geqslant \mathbf{1}$ and $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^t$ with $x_i \in \{0, 1\}$. For example, for the graph $H$ in Fig. 2, $\gamma(H) = 2$ and $\boldsymbol{x} = (0, 1, 0, 0, 0, 1)^t$ is the characteristic function of the dominating set $\{v_2, v_6\}$. In general a function $f : V \to \mathscr{P}$ may clearly be thought of as a vector $\boldsymbol{f}$ in $\mathscr{P}^n$. We say that $\boldsymbol{f}$ is a $\mathscr{P}$-dominating vector if and only if $N\boldsymbol{f} \geqslant \mathbf{1}$.

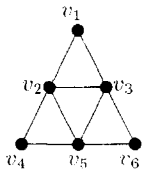


Fig. 2. The Hajós graph $H$.

Let $\mathscr{P}$ be a subset of the reals $\mathbf{R}$. We say a function $f: V \to \mathscr{P}$ is an *efficient $\mathscr{P}$-dominating function* if for every vertex $v$ it holds that $f[v] = 1$. Equivalently, $Nf = \mathbf{1}$ where $\mathbf{1}$ denotes the all 1's vector in $\mathbf{R}^n$. For example, consider the graph shown in Fig. 2. The function that assigns 1 to vertices $v_2$, $v_3$ and $v_5$, and $-1$ to vertices $v_1$, $v_4$, and $v_6$, is an efficient $\{-1, 1\}$-dominating function.

Bange et al. [1] established a conjecture of McRae that all efficient $\mathscr{P}$-dominating functions for a graph have the same weight. We provide a simple proof of this result.

**Theorem 1** (Bange et al. [1]). *If $f_1$ and $f_2$ are any two efficient $\mathscr{P}$-dominating functions for a graph $G$, then $w(f_1) = w(f_2)$.*

**Proof.** Since $N$ is symmetric,

$$f_1^t N f_2 = (f_1^t N) f_2 = (N f_1)^t f_2 = \mathbf{1}^t f_2 = w(f_2),$$

and

$$f_1^t N f_2 = f_1^t (N f_2) = f_1^t \mathbf{1} = w(f_1).$$

Thus $w(f_1) = w(f_2)$, as required. □

A function is *nonnegative* if all the function values are nonnegative. We denote a function which is both nonnegative and efficient $\mathscr{P}$-dominating as an *NE$\mathscr{P}$D-function*. For example, if $G$ is a regular graph of degree $r$, then the function $f$ that assigns to each vertex the value $1/(r+1)$ is an NE$\mathscr{P}$D-function for $G$. If $G$ is a complete bipartite graph of order at least 3 with one partite set $\mathscr{L}$ of cardinality $l$ and the other $\mathscr{R}$ of cardinality $r$, then the function $f$ that assigns to each vertex of $\mathscr{L}$ the value $(r-1)/(lr-1)$ and to each vertex of $\mathscr{R}$ the value $(l-1)/(lr-1)$ is an NE$\mathscr{P}$D-function for $G$.

Grinstead and Slater [17] called a graph which has an NE**R**D-function 'fractionally efficiently dominatable'. Using linear programming duality they observed that if a graph $G$ has a NE**R**D-function $g$ then $\gamma_f(G) = w(g)$. We use related ideas to show that the property of possessing an NE**R**D-function is the key to the real domination number of a graph. (Here $\mathbf{Q}$ denotes the rationals.)

**Theorem 2.** *For any graph $G$,*

$$\gamma_{\mathbf{R}}(G)=\gamma_{\mathbf{Q}}(G)=\begin{cases} w(f) & \text{if } G \text{ has an } NE\mathbf{Q}D\text{-function } f, \\ -\infty & \text{otherwise.} \end{cases}$$

**Proof.** Of course, by linear algebra, if a graph has a NE**R**D-function then it has an NE**Q**D-function. The concept of real domination can be formulated in terms of solving the following linear programming problem:

*Real Domination* $\gamma_{\mathbf{R}}(G)$

$$\min\ \mathbf{1}^t\boldsymbol{x}(=\min \textstyle\sum_{i=1}^n x_i) \quad \text{s.t.} \begin{cases} N\boldsymbol{x}\geqslant \mathbf{1} \\ x_i \text{ unrestricted.} \end{cases}$$

*Dual*

$$\max\ \mathbf{1}^t\boldsymbol{y}(=\max \textstyle\sum_{i=1}^n y_i) \quad \text{s.t.} \begin{cases} N\boldsymbol{y}=\mathbf{1} \\ y_i\geqslant 0. \end{cases}$$

The dual of the above linear programming problem is shown. Since the min problem has a feasible solution (simply take the characteristic function of any dominating set), there are only two possible categories into which solutions to the max and min problems can fall: (1) both problems have feasible solutions, in which case both objective functions have the same solutions; and (2) the max problem has no feasible solution, in which case the objective function for the min problem is unbounded below.

If (1) holds, then the max problem has a feasible solution. However, every feasible solution to the max problem corresponds to an NE**R**D-function for the graph $G$. Hence, the solution to the max problem is an NE**R**D-function of maximum weight. However, by Theorem 1, all NE**R**D-functions have the same weight, so the solution to the max problem (and therefore the min problem) is $w(f)$, where $f$ is an arbitrary NE**R**D-function for $G$; so $\gamma_{\mathbf{R}}(G)=w(f)$. If (2) holds, then $\gamma_{\mathbf{R}}(G)=-\infty$. □

It follows from the theorem that, for any subset $\mathscr{P}$ of $\mathbf{R}$, if a graph $G$ has an NE$\mathscr{P}$D-function $f$, then $\gamma_{\mathscr{P}}(G)=\gamma_{\mathbf{R}}(G)=w(f)$, since $f$ is an NE**R**D-function of $G$, and so $w(f)\geqslant\gamma_{\mathscr{P}}(G)\geqslant\gamma_{\mathbf{R}}(G)=w(f)$.

Also, if there exists a dominating function with total weight less than 1 the **R**-domination number is $-\infty$. For example, we observed earlier that there is a $\{-1,1\}$-dominating function of total weight 0 for the graph $H$ shown in Fig. 2. So $\gamma_{\mathbf{R}}(H)=-\infty$.

## 3. Efficient dominating sets

When $\mathscr{P}=\{0,1\}$ an efficient $\mathscr{P}$-dominating function of a graph $G$ is the characteristic function of a so-called *efficient dominating set* $D$ of $G$: $|N[v]\cap D|=1$ for every $v\in V$. (Equivalently, $D$ dominates $G$ and $u,v\in D$ implies $d(u,v)\geqslant 3$.) Efficient dominating sets were introduced by Bange et al. [2,3].

If a graph $G$ has an efficient dominating set $D$, then, for $\{0,1\}\subseteq\mathscr{P}$, $G$ has an NE$\mathscr{P}$D-function (simply take the characteristic function of $D$). However the converse is not true. Many graphs that do not have efficient dominating sets will have an

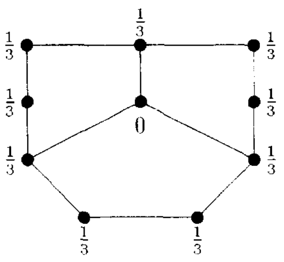


Fig. 3. A NE**R**D-function.

NE**R**D-function. For example, the graph $G$ shown in Fig. 3 has an nonnegative efficient $[0,1]$-dominating function as illustrated, but does not have an efficient dominating set.

Another consequence of the above result is that if $\{0,1\} \subseteq \mathscr{P} \subseteq \mathbf{R}$, and graph $G$ has an efficient dominating set, then $\gamma_{\mathscr{P}}(G) = \gamma(G)$. This follows as the characteristic function of the efficient dominating set is an NE$\mathscr{P}$D-function for $G$. The converse, however, is not true. For example, the graph $G$ shown in Fig. 3 has an NE$\mathscr{P}$D-function $f$ as illustrated, and so $\gamma_f(G) = w(f) = 3$. Furthermore, it is evident that $\gamma(G) = 3$. However, the graph $G$ does not possess an efficient dominating set. Hence, $\gamma_{\mathbf{R}}(G) = \gamma(G)$ does not necessarily imply that $G$ has an efficient dominating set.

However, for trees this does follow:

**Theorem 3.** *For any tree $T$, $T$ has a NE**R**D-function if and only if it has an efficient dominating set.*

**Proof.** For any graph, if it has an efficient dominating set, then it has an NE**R**D-function. So we need to establish the converse. We will need the following lemma.

**Lemma 4.** *Let $T$ be a tree and $f$ a nonnegative function of $T$ such that for every vertex $v$, $f[v] \leqslant 1$. If $f[u] = 1$ for a leaf $u$ of $T$, then every vertex at distance 2 from $u$ of $T$ has weight 0 under $f$.*

**Proof.** Suppose $u$'s neighbour is $v$ and let $S = N(v) - \{u\}$. Then $1 \geqslant f[v] = f[u] + f(S) = 1 + f(S)$, so $f(S) = 0$. But $f$ is nonnegative, so $f(w) = 0$ for all $w \in S$. $\square$

We now prove by induction the statement that if a tree $T = (V, E)$ has a NE**R**D-function $f$, then it has an efficient dominating set $S$ such that $S \subseteq \{v \in V \mid f(v) > 0\}$.

We proceed by induction on the number $m$ of edges in the tree. The base case when the tree is empty or has maximum degree 1 is trivial. So, assume that for all trees $T' = (V', E')$ with less than $m$ edges that if $T'$ has a NE**R**D-function $f$, then it has an efficient dominating set $S$ such that $S \subseteq \{v \in V' \mid f(v) > 0\}$. Let $T = (V, E)$ be a tree on $m$ edges with diameter $d$ that has a NE**R**D-function $f$.

If $d=2$, then $T$ is a star $K_{1,m}$. By Lemma 4, the central vertex $u$ has weight 1 and the leaves have weight 0 under $f$. Letting $S=\{u\}$ we have an efficient dominating set $S$ satisfying $S \subseteq \{v \in V \mid f(v)>0\}$. If $d=3$, then it follows from Lemma 4 that $T$ is a path $P_4$ with the two central vertices of weight 0 and the two leaves of weight 1 under $f$. Letting $S$ consist of the two leaves, we have an efficient dominating set $S$ satisfying $S \subseteq \{v \in V \mid f(v)>0\}$.

For $d \geqslant 4$, consider a longest path in $T$ and let $v$ and $w$ be the third and fourth vertices on that path. The removal of the edge $vw$ yields two trees $T_v$ (containing $v$) and $T_w$. By Lemma 4, $f(v)=0$. So the restriction of $f$ to $T_w$ is a NERD-function of $T_w$. Hence, by induction, there exists an efficient dominating set $S_w$ of $T_w$ such that $S_w \subseteq \{v \in V \mid f(v)>0\}$. It remains to extend $S_w$ to the desired dominating set of $T$.

If $v$ has a neighbour $x$ in $T_v$ with weight 1 under $f$, then every other neighbour of $v$, including $w$, has weight 0 under $f$. So $w \notin S_w$ and if $v$ has degree 3 or more, each neighbour of $v$ other than $x$ and $w$ has weight 0. Since any such neighbour must be dominated by an adjacent endvertex $z$ with $f[z]=1$, the neighbours must have degree 2 with $f(z)=1$. These endvertices together with $x$ and $S_w$ form the desired efficient dominating set.

If every neighbour of $v$ in $T_v$ has weight less than 1 under $f$, then all the neighbours of $v$ in $T_v$ have degree exactly 2 and all the leaves in $T_v$ have positive weight under $f$. If $w \in S_w$, then let $S$ be $S_w$ together with all the leaves of $T_v$. If $w \notin S_w$, then since $w$ is dominated by $S_w$, one of its neighbours in $T_w$ must have positive weight under $f$ and hence $f(w)<1$. From this it follows that at least one neighbour $x$ of $v$ in $T_v$ has positive weight under $f$. So let $S$ be $S_w \cup \{x\}$ together with the leaves of $T_v$ which are nonadjacent to $x$. In both cases we produce an efficient dominating set $S$ of $T$ satisfying $S \subseteq \{v \in V \mid f(v)>0\}$. □

For what other classes of graphs does the existence of a NERD-function imply the existence of an efficient dominating set? Consider for example chordal graphs. A graph is *chordal* if it contains no cycle of length greater than three as an induced subgraph. A *strongly chordal graph* is a chordal graph that contains no induced trampoline, where a *trampoline* consists of a $2n$-cycle $v_1 v_2 \ldots v_{2n} v_1$ in which the vertices $v_{2i}$ of even subscript form a complete graph on $n$ vertices.

Since Farber [13] showed that for strongly chordal graphs their fractional domination number is equal to their domination number, it follows from Theorem 2 and the above discussion that:

**Theorem 5.** *For any strongly chordal graph $G$,*

$$\gamma_{\mathbf{R}}(G)=\begin{cases} \gamma(G) & \text{if } G \text{ has an NERD-function,} \\ -\infty & \text{otherwise.} \end{cases}$$

But it remains unresolved whether in (strongly) chordal graphs the existence of a NERD-function implies the existence of an efficient dominating set.

## 4. Products

Let $G=(V,E)$ and $H=(V',E')$ be two graphs with disjoint vertex sets. The *cartesian product* $G \times H$ has vertex set $V \times V'$ and two vertices $(a,b)$ and $(c,d)$ are adjacent if either $a=c$ and $bd \in E'$ or $b=d$ and $ac \in E$. The *strong direct product* $G \cdot H$ has vertex set $V \times V'$, and two vertices $(a,b)$ and $(c,d)$ are adjacent if $c \in N[a]$ and $d \in N[b]$.

Fisher et al. [15] observed that for all graphs $G$ and $H$, $\gamma_f(G \cdot H)=\gamma_f(G)\gamma_f(H)$. From this they deduced that

$$\gamma_f(G \times H) \geqslant \gamma_f(G)\gamma_f(H),$$

which established the fractional version of Vizing's conjecture.

For real domination there is no result analogous to his conjecture. For example, $P_3$ has an efficient dominating set, but $P_3 \times P_3$ does not have an NE**R**D-function. So $\gamma_{\mathbf{R}}(P_3 \times P_3)=-\infty$ while $\gamma_{\mathbf{R}}(P_3)=1$.

However, the result on strong direct products does generalise. For, if $g$ and $h$ are dominating functions of $G$ and $H$ respectively, then consider the function $f : V \times V' \to \mathbf{R}$ defined by $(a,b) \mapsto g(a)h(b)$ (and denoted by $f=g \otimes h$). Since $N[(a,b)]=N[a] \times N[b]$, the function $f$ is a dominating function of $G \cdot H$, and if $g$ and $h$ are both efficient then so is $f$. Furthermore, $w(f)=w(g) \times w(h)$.

**Theorem 6.** *For all graph $G$ and $H$*

$$\gamma_{\mathbf{R}}(G \cdot H)=\begin{cases} \gamma_{\mathbf{R}}(G) \cdot \gamma_{\mathbf{R}}(H) & \text{if } \gamma_{\mathbf{R}}(G) \text{ and } \gamma_{\mathbf{R}}(H) \text{ both positive,} \\ -\infty, & \text{otherwise.} \end{cases}$$

**Proof.** If $G$ and $H$ have NE**R**D-functions $g$ and $h$, then $f=g \otimes h$ is an NE**R**D-function for $G \cdot H$ and by Theorem 2, $\gamma_{\mathbf{R}}(G \cdot H)=w(f)=w(g) \cdot w(h)=\gamma_{\mathbf{R}}(G) \cdot \gamma_{\mathbf{R}}(H)$.

On the other hand, suppose $\gamma_{\mathbf{R}}(G)$ say is $-\infty$. Then let $h$ be any dominating function of $H$ with positive total weight (for example the all 1s function). Then since $G$ has a **R**-dominating function $g$ with arbitrarily negative weight, so does $G \cdot H$: namely, $g \otimes h$. This means that $\gamma_{\mathbf{R}}(G \cdot H)=-\infty$. □

Related results are discussed in [14].

## 5. Hardness results

If $\mathscr{P}=\mathbf{R}$ or $\mathbf{Q}$, then the determination of $\gamma_{\mathscr{P}}(G)$ can be formulated in terms of solving a linear programming problem, and so can be computed in polynomial-time (see, e.g., [22]). On the other hand, the determination of the domination, signed domination and minus domination numbers has been shown to be NP-complete even when restricted to bipartite graphs (see [7,12,18]) and chordal graphs (see [4,12,18]). In this section we show that the problem is NP-hard provided $\mathscr{P}$ contains 0 and 1 and is bounded from above.

The following decision problem for the domination number of a graph is known to be NP-complete (see [23]). (A cubic graph is 3-regular.)

`Dominating Set` ($DM$)

**Instance:** A cubic (planar) graph $G=(V,E)$ and a positive integer $k$.

**Question:** Does $G$ have a dominating set of cardinality $k$ or less?

In this section we consider the general version:

$\mathscr{P}$-`Dominating Function` ($\mathscr{P}DM$)

**Instance:** A graph $H=(V,E)$ and a positive integer $j$.

**Question:** Does $H$ have a $\mathscr{P}$-dominating function of weight $j$ or less?

We will demonstrate a polynomial-time reduction of the normal domination problem to the $\mathscr{P}$-domination problem. To do this, we introduce some notation. We define a *pendant* in a graph $G$ as a subgraph on four vertices, three of which have degree 2 in $G$, that induce a 4-cycle. Equivalently, a pendant is an end-block that is a 4-cycle. The vertex of the pendant of degree more than 2 in $G$ we call the '*attacher*' and the vertex not adjacent to the attacher we call the '*dangler*'.

**Lemma 7.** *Let $m$ be a positive integer and let $\mathscr{P}=\{i\in\mathbf{Z}\mid i\leqslant m\}$. For any graph $G$, there exists a $\mathscr{P}$-dominating function of weight $\gamma_{\mathscr{P}}(G)$ such that for each pendant the two neighbours of the attacher have the same weight as the attacher.*

**Proof.** Among all $\mathscr{P}$-dominating functions of $G$ of minimum weight, let $f:V\to\mathscr{P}$ be one for which the sum of the weights of the danglers is as small as possible. We show that $f$ satisfies the requirements of the lemma. Let $abcda$ be a pendant with attacher $a$ and dangler $c$.

Suppose $f[c]>1$. If $f(a)<m$, then decrement the weight of $c$ and increment the weight of $a$. The result is a $\mathscr{P}$-dominating function that contradicts the choice of $f$. If $f(a)=m$, then it follows that $f[b]\geqslant f[c]>1$ and $f[d]>1$ so that one can decrement the weight of $c$ and still have a $\mathscr{P}$-dominating function, once again contradicting the choice of $f$. Thus $f[c]=1$.

Suppose $f[b]>1$. If $f(d)<m$, then we may decrement the weight of $c$ and increment the weight of $d$ to obtain a $\mathscr{P}$-dominating function that contradicts the choice of $f$. If $f(d)=m$, then $f[c]=f(b)+f(c)+m\geqslant f[b]>1$, which is a contradiction. Hence $f[b]=1$. Similarly, $f[d]=1$. It follows that $f(a)=f(b)=f(d)$ and $f(c)=1-2f(a)$. $\square$

**Theorem 8.** *Let $m$ be a positive integer and let $\{0,1\}\subseteq\mathscr{P}\subseteq\{i\in\mathbf{Z}\mid i\leqslant m\}$. Then $\mathscr{P}$-*`Dominating Function` *is* NP-*complete (even for planar graphs).*

**Proof.** It is obvious that $\mathscr{P}$DM is a member of NP since we can, in polynomial time, guess at a function $f:V\to\mathscr{P}$ and verify that $f$ has weight at most $j$ and is a $\mathscr{P}$-dominating function. We next show how a polynomial-time algorithm for $\mathscr{P}$DM could be used to solve DM in polynomial time. Given a cubic graph $G=(V,E)$ on $n$ vertices, and a positive integer $k$, construct the graph $H=(V',E')$ by attaching $x=\lceil(3m-1)/2\rceil$ pendants to each vertex of $G$. It is easy to see that the construction of the graph $H$ can be accomplished in polynomial time, and that $H$ is planar if $G$ is. Let $\mathscr{Q}=\{i\in\mathbf{Z}\mid i\leqslant m\}$.

**Lemma 9.** $\gamma_{\mathscr{Q}}(H)=\gamma_{\mathscr{P}}(H)=\gamma(H)=nx+\gamma(G)$.

**Proof.** Let $D$ be a minimum dominating set of $G$, and let $D'$ be the extension of $D$ to $H$ that includes the dangler of each pendant. Then $D'$ is a dominating set of $H$ of cardinality $nx+\gamma(G)$. Hence $\gamma_{\mathscr{Q}}(H)\leqslant\gamma_{\mathscr{P}}(H)\leqslant\gamma(H)\leqslant nx+\gamma(G)$.

To show that $nx+\gamma(G)$ is a lower bound on $\gamma_{\mathscr{Q}}(H)$, let $f$ be a $\mathscr{Q}$-dominating function of $H$ that satisfies the requirements of Lemma 7. Suppose one of the attachers $a$ has a negative weight. Then, by Lemma 7, all its $2x$ neighbours in its pendants have negative weight. Thus, since $G$ is cubic, $f[a]\leqslant -1\cdot(2x+1)+3m\leqslant 0$, which produces a contradiction. Hence, $f$ assigns to all the attachers a nonnegative weight. If $f(a)=0$, then, by Lemma 1, all its neighbours in the pendants also have weight 0. So one of its neighbours in $G$ must have positive weight. This means that the attachers with positive weight form a dominating set of $G$. However the proof of Lemma 7 shows that the closed neighbourhood sum of each dangler is 1. So the total weight of $f$ is at least $nx+\gamma(G)$. □

The above lemma implies that if we let $j=nx+k$, then $\gamma(G)\leqslant k$ if and only if $\gamma_{\mathscr{P}}(H)\leqslant j$. This completes the proof of Theorem 8. □

Consider $\mathscr{P}$-domination when $\mathscr{P}=\mathbf{Z}$. If $\gamma_{\mathbf{Q}}(G)=-\infty$ then $\gamma_{\mathbf{Z}}(G)=-\infty$; this follows since, if one takes a $\mathbf{Q}$-dominating function $f$ and multiplies all the weights by the least common multiple of the weights' denominators, one obtains a $\mathbf{Z}$-dominating function. So if there is a $\mathbf{Q}$-dominating function of arbitrarily negative weight then there is a such a $\mathbf{Z}$-dominating function too.

However, it remains an open problem to determine the complexity of $\mathbf{Z}$-domination. Also we do not know of a graph-theoretic proof that shows that $\mathbf{Z}$-domination is a member of NP. This is, however, a consequence of the general result that integer programming is in NP (see [5]).

## 6. Integer intervals

In this section, we consider as weight set an interval of integers. Specifically, we let $\mathscr{P}=\{A,A+1,\dots,B\}$, where $A$ and $B$ are integers satisfying $1\leqslant B\leqslant 1-A$. For example, $A=-1$ and $B=1$ correspond to a minus dominating function. We showed in the previous section that the $\mathscr{P}$-domination problem is NP-complete for general graphs.

Here we present a linear-time algorithm for finding a minimum $\mathscr{P}$-dominating function in a tree $T$. The algorithm is based on the one for minus domination given in [12].

Let $T_r = (V, E, r)$ be a rooted tree with root $r$ and $v$ a vertex of $T$. Then the *level number* $\ell(v)$ of $v$ is the length of the unique $r$-$v$ path in $T$. The maximum of the level numbers of the vertices of $T$ is called the *height* of $T$ and is denoted by $h(T)$. If a vertex $v$ of $T$ is adjacent to $u$ and $\ell(u) > \ell(v)$, then $u$ is called a *child* of $v$; if the level numbers of the vertices on the $v$-$w$ path are monotonically increasing, then $w$ is a *descendant* of $v$. The subtree of $T$ induced by $v$ and all of its descendants is called the *maximal subtree* of $T$ rooted at $v$.

We introduce the following notation. For a rooted tree $T_r = (V, E, r)$ with root $r$, we call a function $f : V \to \mathscr{P}$ an *almost $\mathscr{P}$-dominating function* if the sum of its function values over every closed neighbourhood except that of the root is at least 1, and the closed neighbourhood sum of the root is at least $1 - B$. We define $\gamma_{a\mathscr{P}}(T_r)$ to be the minimum weight $w(f)$ of an almost $\mathscr{P}$-dominating function $f$ of the tree $T_r$. Any such $f$ is called a minimum almost $\mathscr{P}$-dominating function. Note that if $f$ is an almost $\mathscr{P}$-dominating function of $T_r$ then $f$ restricted to any maximal subtree is one also.

Furthermore, we define $\gamma'_{a\mathscr{P}}(T_r, \alpha)$ to be the minimum weight of an almost $\mathscr{P}$-dominating function of the tree $T_r$ with $r$ receiving weight $\alpha$. We define $\gamma''_{a\mathscr{P}}(T_r, \theta)$ to be the minimum weight of an almost $\mathscr{P}$-dominating function of the tree $T_r$ rooted at $r$ with the sum of the values assigned to the vertices in the closed neighbourhood of $r$ receiving weight $\theta$.

The following recursive linear-time algorithm finds a minimum almost $\mathscr{P}$-dominating function in a tree $T$. The vertices of $T$ are assigned values from the weight set $\mathscr{P}$ starting with the vertices at the highest level and ending with the root at the lowest level. Hence a vertex $v$ receives a weight only once all its descendants have received weights. For each vertex $v$, the algorithm associates a variable *ChildSum*$(v)$ which is the sum of the values assigned to the children of $v$.

**Algorithm 1. $\mathscr{P}$-domination of tree:**
**Input**: A rooted tree $T = (V, E)$ on $n$ vertices with the vertices labelled from 1 to $n$ such that $label(w) > label(y)$ if the level of vertex $w$ is less than the level of vertex $y$.

**For** $i \leftarrow 1$ to $n$ **do**
(1) *ChildSum*$(i) \leftarrow$ (sum of weights of the children of vertex $i$).
(2) **If** *ChildSum*$(i) < 1 - 2B$ **then**
- Increase the weights of the children of vertex $i$ (so that each weight remains at most $B$) until *ChildSum*$(i) = 1 - 2B$.
- $f(i) \leftarrow B$.

(3) **If** *ChildSum*$(i) \geqslant 1 - 2B$ **then**
- let $f(i)$ be the minimum weight that may be assigned to vertex $i$ that yields an almost $\mathscr{P}$-dominating function on the maximal subtree rooted at $i$.

**end for**

We now verify the validity of Algorithm 1.

**Theorem 10.** *Let $T_r = (V, E, r)$ be a rooted tree of order $n$ with root $r$, and let $f$ be the function produced by Algorithm 1. Then the following five conditions hold:*
(1) *The function $f$ is a minimum almost $\mathscr{P}$-dominating function of the rooted tree $T_r$; so $\gamma_{a.\mathscr{P}}(T_r) = w(f)$.*
(2) *The root $r$ receives the maximum weight $M$ over all minimum almost $\mathscr{P}$-dominating functions.*
(3) *The closed neighbourhood sum of $r$ is the maximum value $S$ over all minimum almost $\mathscr{P}$-dominating functions.*
(4) $\gamma'_{a.\mathscr{P}}(T_r, M+\alpha) = \gamma_{a.\mathscr{P}}(T_r) + \alpha$ *(for $0 \leqslant \alpha \leqslant B - M$).*
(5) $\gamma''_{a.\mathscr{P}}(T_r, S+\theta) = \gamma_{a.\mathscr{P}}(T_r) + \theta$ *(for $0 \leqslant \theta \leqslant B \cdot (\deg r) - S$).*

Of course, Conditions (4) and (5) imply Conditions (2) and (3), respectively.

**Proof.** We proceed by induction on the height $h$ of the rooted tree. If $h = 0$, then $T_r$ is the trivial tree, and so $Childsum(r) = 0$ and the root $r$ will be assigned the value $1 - B$ in Step 3. The five conditions listed above are easily checked. Assume, then, that for all rooted trees of height at most $h$, where $h \geqslant 0$, that the five conditions are satisfied, and let $T_r = (V, E, r)$ be a rooted tree with root $r$ of height $h + 1$.

Let $N(r) = \{v_1, v_2, \ldots, v_s\}$ be the set of children of the root $r$. For each $i = 1, \ldots, s$, let $T_i$ be the maximal subtree of $T$ rooted at $v_i$ and let $f_i$ be the function produced by Algorithm 1 on the subtree $T_i$. Let $f : (V - \{r\}) \to \mathscr{P}$ be the function defined by $f(x) = f_i(x)$ for each $x$ in $T_i$ ($i = 1, \ldots, s$). Then $w(f) = \sum_{i=1}^{s} w(f_i)$. By the inductive hypothesis, it follows that the total weight of any almost $\mathscr{P}$-dominating function on $T_r$ where the root $r$ receives a weight of $m$ is at least $w(f) + m$. We consider two possibilities.

*Case* 1. *The function $f$ can be extended to an almost $\mathscr{P}$-dominating function by assigning to the root the weight $B$.*

Let $M$ denote the minimum weight that may be assigned to the root $r$ which extends $f$ to an almost $\mathscr{P}$-dominating function $f'$. Then $\gamma_{a.\mathscr{P}}(T_r) \leqslant w(f') = w(f) + M$. Let $g$ be any almost $\mathscr{P}$-dominating function of $T_r$ where the root $r$ receives a weight $m < M$. By the choice of $M$ the value assigned to the root $r$ in $f'$ must be the smallest it can be because of the neighbourhood sum of either $r$ or one of its children. Suppose $M$ was critical to $r$'s neighbourhood sum. Then in $g$ the total weight of $r$'s children must be at least $M - m$ more than it is in $f$. But by (4) that means that the total weight in $g$ of the descendants of $r$ is at least $M - m$ more than it is in $f$; and so $w(g) \geqslant w(f')$. Suppose $M$ was critical to the neighbourhood sum of some child $x$ of $r$. Then in $g$ $x$'s closed neighbourhood excluding $r$ has at least $M - m$ more weight. But by Condition (5) that means that the weight of $g$ on the maximal subtree rooted at $x$ is at least $M - m$ more than it is in $f$, so, once again, it follows that $w(g) \geqslant w(f')$. Hence $\gamma_{a.\mathscr{P}}(T_r) = w(f') = w(f) + M$. Conditions (2) and (4) follow since $\gamma'_{a.\mathscr{P}}(T_r, M+\alpha) \geqslant w(f) + (M + \alpha)$. Conditions (3) and (5) follow since any increase in the closed neighbourhood of $r$ costs that increase.

*Case* 2. *The function $f$ cannot be extended to an almost $\mathscr{P}$-dominating function of $T_r$.*

By the inductive hypothesis, $f_i$ is an almost $\mathscr{P}$-dominating function for $T_i$. Hence if we extend $f$ to $V(T_r)$ by assigning to the root the weight $B$, then the closed neighbourhood sum of every vertex in $T_r$ different from the root $r$ is positive. It follows that, if we let $C = \sum_{i=1}^{s} f(v_i)$, we must have $C < 1 - 2B$. Furthermore, by Condition (2), the children of $r$ have the maximum weight they can have and any increase in their weights requires weight equal to the increase. So if $g$ is an almost $\mathscr{P}$-dominating function, then the weight of $g$ on $r$'s descendants is at least $1 - 2B - C$ more than the weight of $f$ on $r$'s descendants, as $r$'s children receive at least $1-2B-C$ more weight in $g$ than in $f$. Hence $\gamma_{a\mathscr{P}}(T_r) \geqslant w(f)+1-B-C$. But the algorithm does find an almost $\mathscr{P}$-dominating function with this weight. Thus $\gamma_{a\mathscr{P}}(T_r) = w(f) + 1 - B - C$. Further, the same reasoning implies that Conditions (3) and (5) are true. By the algorithm, the root is assigned the weight $B$, so (2) is true as is (4) vacuously. □

Now all that remains is to compute the $\mathscr{P}$-domination number of the rooted tree $T_r$ with root $r$. Let $f$ be the function produced by Algorithm 1, and let $f[r] = S$. We know by Theorem 10 that $S \geqslant 1 - B$. If $S \geqslant 1$, then it follows from Condition (1) of Theorem 10 that $f$ is also a minimum $\mathscr{P}$-dominating function of $T_r$. If $S \leqslant 0$, then we increase the sum of the weights of the vertices in the closed neighbourhood of $r$ by $1 - S$ (so that each weight remains at most $B$) to produce a $\mathscr{P}$-dominating function $f'$ of $T_r$ of weight $w(f) + 1 - S$. Hence $\gamma_{\mathscr{P}}(T_r) \leqslant w(f) + 1 - S$. However, it follows from Conditions (3) and (5) of Theorem 10 that $\gamma_{\mathscr{P}}(T_r) \geqslant \gamma_{a\mathscr{P}}(T_r)+1-S = w(f)+1-S$. Consequently, $f'$ is a minimum $\mathscr{P}$-dominating function of $T_r$ (of weight $w(f) + 1 - S$).

## 7. Other $\mathscr{P}$-parameters

One can define $\mathscr{P}$-analogues of other graphical parameters. In some cases, such as independence number and total domination, one obtains the analogue of Theorem 2; in other cases, such as upper domination, there is an even simpler characterisation of the real version. We discuss here only the independence and upper domination numbers.

Let $G = (V, E)$ be a graph where $|V| = n$ and $|E| = m$. Let $I$ denote the $n \times m$ incidence matrix of $G$. We say a function $f : V \to \mathscr{P}$ is a *$\mathscr{P}$-independence function* of $G$ if for every edge $e$ the sum of the values (weights) assigned under $f$ to the two ends of $e$ is at most 1. The *$\mathscr{P}$-independence number* $\beta_{\mathscr{P}}(G)$ of $G$ is defined to be the supremum of $w(f)$ taken over all $\mathscr{P}$-independence functions $f$. If $\mathscr{P} = \{0, 1\}$ then one obtains the normal independence number.

An obvious lower bound on $\beta_{\mathbf{R}}$ is $n/2$ attained by assigning to every vertex a weight of $1/2$. We say a function $g : E \to \mathscr{P}$ is an *efficient $\mathscr{P}$-matching function* of $G$ if for every vertex $v$ the sum of the values assigned under $g$ to the edges incident with $v$ is 1. One can obtain a result similar to Theorem 2.

**Theorem 11.** *For any graph G on n vertices,*

$$\beta_{\mathbf{R}}(G) = \beta_{\mathbf{Q}}(G) = \begin{cases} n/2 & \text{if } G \text{ has a nonnegative efficient } \mathbf{R}\text{-matching function,} \\ +\infty & \text{otherwise.} \end{cases}$$

**Proof.** We use the same argument as in the proof of Theorem 2. The real independence problem is to max $\mathbf{1}'\boldsymbol{x}$ subject to $I'\boldsymbol{x} \leqslant \mathbf{1}$ and $\boldsymbol{x}$ unrestricted. The linear-programming dual is to min $\mathbf{1}'\boldsymbol{y}$ subject to $I\boldsymbol{y} = \mathbf{1}$ and $\boldsymbol{y} \geqslant 0$. Since the max problem is always feasible, there are only two possibilities. If the min problem is feasible then the solution to the max is the solution to the min, but as in Theorem 1 all efficient matching vectors have the same weight. If the min problem is not feasible then the solution to the max is $+\infty$. $\square$

The upper domination number is defined as the cardinality of the largest minimal dominating set of $G$ (see for example, [6]). We say a $\mathscr{P}$-dominating function $f$ is a *minimal $\mathscr{P}$-dominating function* if there does not exist a $\mathscr{P}$-dominating function $h$, $h \neq f$, such that $h(v) \leqslant f(v)$ for every $v \in V$. The *upper $\mathscr{P}$-domination number* for $G$ is $\Gamma_{\mathscr{P}}(G) = \sup\{f(V) \mid f : V \to \mathscr{P}$ is a minimal $\mathscr{P}$-dominating function on $G\}$.

**Theorem 12.** *Let $\mathscr{P} = \boldsymbol{Z}$, $\boldsymbol{Q}$ or $\boldsymbol{R}$. Then for any connected graph $G = (V, E)$,*

$$\Gamma_{\mathscr{P}}(G) = \begin{cases} 1 & G \text{ is complete,} \\ +\infty & \text{otherwise.} \end{cases}$$

**Proof.** We consider only the case where $\mathscr{P} = \boldsymbol{Z}$. (The same proof works for $\Gamma_{\boldsymbol{R}}$ or $\Gamma_{\boldsymbol{Q}}$.) The result is obvious if $G$ is complete. So assume $G$ is not complete. Let $v_1$ be a vertex of maximum degree in $G$. Then among all the vertices in $V - N[v_1]$, let $v_2$ be chosen to have maximum degree in $G$. If $V - (N[v_1] \cup N[v_2]) \neq \emptyset$, then among all the vertices in $V - (N[v_1] \cup N[v_2])$, let $v_3$ be chosen to have maximum degree in $G$. Continuing in this way, we may construct a maximal independent set $A = \{v_1, v_2, \ldots, v_a\}$ of vertices the union of whose closed neighbourhoods is the set $V$. Let $B = V - A$. Let $M$ be a positive integer and define an assignment $f$ by

$$f(v) = \begin{cases} -M & \text{if } v \in B, \\ M \cdot \deg v + 1 & \text{if } v \in A. \end{cases}$$

If $v \in B$, then $v$ is adjacent to some vertex of $A$; let $v_i$ be the vertex of smallest index in $A$ that is adjacent to $v$. Then by our choice of $v_i$, we know that $\deg v_i \geqslant \deg v$. It follows that $f[v] \geqslant -M \cdot \deg v + (M \cdot \deg v_i + 1) \geqslant 1$. Therefore, $f[v] \geqslant 1$ for all $v \in B$. If $v \in A$, then $f[v] = 1$. Since $\bigcup_{v \in A} N[v] = V$ it follows that $f$ is a minimal real dominating assignment. The weight of $f$ is given by

$$f(V) = \sum_{v \in A} f(v) + \sum_{v \in B} f(v) = |A| + M \cdot |[A, B]| - M \cdot |B|,$$

where $[A,B]$ denotes the set of edges joining $A$ and $B$. Since every vertex of $B$ is adjacent to at least one vertex of $A$, it follows that $|[A,B]| \geqslant |B|$.

If $G$ does not have an efficient dominating set, then at least one vertex of $B$ is adjacent to more than one vertex of $A$, in which case $|[A,B]| > |B|$ and the weight of $f$ tends to $+\infty$ as $M \to \infty$.

Suppose then that each vertex of $B$ is adjacent to a unique vertex of $A$. Thus the sets $N[v_i]$ $(1 \leqslant i \leqslant a)$ form a partition of $V$. Let $S$ be the set of all vertices of $N[v_1]$ of maximum degree (namely, $\deg v_1$) in the induced graph $\langle N[v_1] \rangle$, so $S$ is the set of vertices whose closed neighbourhoods equal the set $N[v_1]$. (Possibly $S$ consists only of $v_1$.) Let $T = N[v_1] - S$. Since $G$ is connected and $G$ is not complete, it is evident that $T \neq \emptyset$. Each vertex of $T$ is not adjacent to at least one other vertex of $T$. Let $t_1$ be a vertex of $T$ of maximum degree in $G$. Let $t_2$ be a vertex of $T - N[t_1]$ of maximum degree in $G$. If $T - (N[t_1] \cup N[t_2]) \neq \emptyset$, then let $t_3$ be a vertex of $T - (N[t_1] \cup N[t_2])$ of maximum degree in $G$. Continuing in this way, we may construct a maximal independent set $U = \{t_1, t_2, \ldots, t_s\}$ $(s \geqslant 2)$ of vertices of $T$. Let $A' = (A - \{v_1\}) \cup U$ and let $B' = V - A'$. Note that $A'$ is an independent set of $G$. We define an assignment $f'$ by

$$f'(v) = \begin{cases} -M & \text{if } v \in B', \\ M \cdot \deg v + 1 & \text{if } v \in A'. \end{cases}$$

Since $U$ is a dominating set of $N[v_1]$, we note that $f'(U) = |U| + M \cdot \sum_{t \in U} \deg t \geqslant |U| + M \cdot |N[v_1] - U|$. Hence if $v \in S$, then $f'[v] = -M \cdot |N[v_1] - U| + f(U) \geqslant |U| \geqslant 2$. If $v \in T - U$, then $v$ is adjacent to some vertex of $U$; let $t_i$ be the vertex of smallest index in $U$ that is adjacent to $v$. Then by our choice of $t_i$, we know that $\deg t_i \geqslant \deg v$. It follows that $f'[v] \geqslant -M \cdot \deg v + (M \cdot \deg t_i + 1) \geqslant 1$. If $v \in B - N[v_1]$, then $f'[v] \geqslant f[v] \geqslant 1$. Therefore, $f'[v] \geqslant 1$ for all $v \in B'$. If $v \in A'$, then $f'[v] = 1$. Since $\bigcup_{v \in A'} N[v] = V$ it follows that $f'$ is a minimal real dominating assignment. The weight of $f'$ is given by

$$f'(V) = \sum_{v \in A'} f'(v) + \sum_{v \in B'} f'(v) = |A'| + M \cdot |[A', B]| - M \cdot |B'|.$$

Since every vertex of $B'$ is adjacent to at least one vertex of $A'$, it follows that $|[A', B']| \geqslant |B'|$. Futhermore, since $v_1 \in B'$ is adjacent to more than one vertex of $A'$, it follows that $|[A', B']| > |B|$ and so the weight of $f'$ tends to $+\infty$ as $M \to \infty$. □

## References


[1] D.W. Bange, A.E. Barkauskas, L.H. Host, P.J. Slater, Generalized domination and efficient domination in graphs, Discrete Math. 159 (1996) 1–11.

[2] D.W. Bange, A.E. Barkauskas, P.J. Slater, Disjoint dominating sets in trees, Sandia Laboratories Report SAND 78-1087J, Sandia Laboratories, Albuquerque, 1978.

[3] D.W. Bange, A.E. Barkauskas, P.J. Slater, Efficient dominating sets in graphs, in: R.D. Ringeisen, F.S. Roberts (Eds.), Applications of Discrete Mathematics, SIAM, Philadelphia, 1988, pp. 189–199.

[4] K.S. Booth, J.H. Johnson, Dominating sets in chordal graphs, SIAM J. Comput. 11 (1982) 191–199.

[5] I. Borosh, L.B. Trebig, Bounds on positive integral solutions of linear Diopantine equations, Proc. Amer. Math. Soc. 55 (1976) 299–304.
[6] E.J. Cockayne, S.T. Hedetniemi, Towards a theory of domination in graphs, Networks 7 (1977) 247–261.
[7] A.K. Dewdney, Fast Turing reductions between problems in NP 4, Report 71, University of Western. Ontario, 1981.
[8] G.S. Domke, G. Fricke, S.T. Hedetniemi, R. Laskar, Relationships between integer and fractional parameters of graphs, in: Graph Theory, Combinatorics, and Applications, vol. 2, Wiley, New York, 1991, pp. 371–387.
[9] G.S. Domke, S.T. Hedetniemi, R. Laskar, Fractional packings, coverings and irredundance in graphs, Congressus Numer. 66 (1988) 227–238.
[10] J.E. Dunbar, S.T. Hedetniemi, M.A. Henning, P.J. Slater, Signed domination in graphs. in: Graph Theory, Combinatorics, and Applications, vol. 1, Wiley, New York, 1995, pp. 311–322.
[11] J.E. Dunbar, S.T. Hedetniemi, M.A. Henning, A.A. McRae, Minus domination in graphs, Discrete Math. 199 (this Vol.) (1999) 35–47.
[12] J.E. Dunbar, W. Goddard, S.T. Hedetniemi, M.A. Henning, A. McRae, On the algorithmic complexity of minus domination in graphs, Discrete Appl. Math. 68 (1996) 73–84.
[13] M. Farber, Domination, independent domination, and duality in strongly chordal graphs, Discrete Appl. Math. 7 (1984) 115–130.
[14] D.C. Fisher, Domination, fractional domination, 2-packing, and graph products, SIAM J. Discrete Math. 7 (1994) 493–498.
[15] D.C. Fisher, J. Ryan, G. Domke, A. Majumdar, Fractional domination of strong direct products, Discrete Appl. Math. 50 (1994) 89–91.
[16] M.R. Garey, D.S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, Freeman, New York, 1979.
[17] D.L. Grinstead, P.J. Slater, Fractional domination and fractional packing in graphs, Congressus Numer. 71 (1990) 153–172.
[18] J.H. Hattingh, M.A. Henning, P.J. Slater, Three-valued $k$-neighbourhood domination in graphs, Australasian J. Combin. 9 (1994) 233–242.
[19] S.M. Hedetniemi, S.T. Hedetniemi, T.V. Wimer, Linear time resource allocation for trees, manuscript, 1987.
[20] S.T. Hedetniemi, R.C. Laskar, Bibliography on domination in graphs and some basic definitions of domination parameters, Discrete Math. 86 (1990) 257–277.
[21] M.A. Henning, Dominating functions in graphs. In: Domination in Graphs: Advanced Topics, Marcel-Dekker (1997) 31–62.
[22] N. Karmarkar, A new polynomial-time algorithm for linear programming, Combinatorica 4 (1984) 373–395.
[23] C.E. Veni-Madhavan, Approximation algorithms for maximum independent set in planar triangle-free graphs, Lecture Notes Comput. Sci., vol. 181, Springer, Berlin, 1984, 381–392.